Understanding the different scaling behavior in various shell models proposed for turbulent thermal convection

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Different scaling behavior has been reported in various shell models proposed for turbulent thermal convection. In this paper, we show that buoyancy is not always relevant to the statistical properties of these shell models even though there is an explicit coupling between velocity and temperature in the equations of motion. When buoyancy is relevant (irrelevant) to the statistical properties, the scaling behavior is Bolgiano-Obukhov (Kolmogorov) plus intermittency corrections. We show that the intermittency corrections of temperature could be solely attributed to fluctuations in the entropy transfer rate when buoyancy is relevant but due to fluctuations in both energy and entropy transfer rates when buoyancy is irrelevant. This difference can be used as a criterion to distinguish whether temperature is behaving as an active or a passive scalar.

PACS numbers: 47.27-i,47.27te

I. INTRODUCTION

Turbulent thermal convection is a problem of great research interest (see, for example, [1, 2] for a review). One interesting issue is to understand the scaling behavior of the velocity and temperature fluctuations. Turbulent thermal convection is often investigated experimentally in Rayleigh-Bénard convection cells, which are closed cells of fluid heated from below and cooled on the top. Such confined turbulent convective flows are highly inhomogeneous as thermal and viscous boundary layers are formed near the top and bottom of the cell. Scaling laws for the central region of such confined turbulent thermal convection have been put forth and shown to be in good agreement with the existing experimental measurements [3]. On the other hand, shell models focussing on the energy cascade process have been studied intensively and proved to be useful for understanding the scaling behavior of velocity fluctuations in inertia-driven turbulence (see, for example, [4] for a review). It is thus natural to also construct shell models for turbulent thermal convection. Shell models are, by construction, boundary-free and thus shell models for turbulent thermal convection are necessarily models of *homogeneous* turbulent thermal convection. It is known that the presence of boundaries generates coherent structures such as plumes and a large-scale mean flow in confined turbulent thermal convection, and these coherent structures can affect the scaling behavior [3]. Thus, scaling behavior in confined turbulent thermal convection and scaling behavior in homogeneous turbulent thermal convection as studied in shell models can be different.

Several shell models for turbulent thermal convection have been proposed and different scaling behavior reported. Specifically, Bolgiano-Obukhov (BO) scaling [5] plus intermittency corrections has been reported in the shell model constructed by Brandenburg [6] and also in the modified model by Suzuki and Toh [7] for some parameter range. On the other hand, Kolmogorov 1941 (K41) scaling [8] plus intermittency corrections has been reported by Jiang and Liu [9] using a shell model extended from the Gledzer-Ohkitani-Yamada (GOY) model [10], which we shall denote as the GOYT model. In this paper, we show that buoyancy is not always significant and directly relevant to the statistical properties even though there is an explicit coupling between velocity and temperature in the equations of motion in all these shell models. We clarify that the two different types of scaling behavior reported correspond respectively to the case when buoyancy is relevant to the statistical properties and the case when it is not. Specifically, the scaling behavior is BO plus intermittency corrections when buoyancy is relevant, and K41 plus intermittency corrections (as one would expect for temperature behaving as a passive scalar) when buoyancy is irrelevant. We show that the intermittency corrections of temperature could be solely attributed to fluctuations in the entropy transfer rate when buoyancy is relevant but due to fluctuations in both energy and entropy transfer rates when buoyancy is irrelevant. This difference might be used as a criterion to distinguish whether temperature is behaving as an active or a passive scalar.

II. SHELL MODELS PROPOSED FOR TURBULENT THERMAL CONVECTION

Two classes of shell models have been proposed for studying turbulent thermal convection. The first class consists of the shell model proposed by Brandenburg [6] and its modified versions [7]. The other class consists of the GOYT model, the shell model extended from the GOY model [9] and the SabraT model [11] from the Sabra model [11]. The Sabra model [12] was proposed to eliminate some undesirable periodic oscillations in the GOY

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model, and have essentially the same scaling behavior as the GOY model. The scaling behavior in the first class of shell models is BO plus corrections in some parameter range while the scaling behavior in the second class of shell models is always K41 plus corrections. In this paper, we focus on two shell models, one from each class. The first one, denoted as the Brandenburg model, is the modified model proposed by Suzuki and Toh [7] without the drag term. The second is the SabraT model.

The basic idea of a shell model is to consider variables in discrete "shells" in Fourier k-space, and construct a set of ordinary differential equations for these variables per shell. For shell models for turbulent thermal convection, there are two variables, the velocity and temperature variables, u_n and θ_n. They can be roughly thought of as the Fourier transforms of the velocity and temperature fields with wavevector $\vec{k}$, whose magnitude satisfies $k_n \le |\vec{k}| \le k_{n+1}$. Here, $k_n = 2^n k_0$ is the wavenumber of the nth shell, with $0 \le n \le N-1$, and $k_0 = 1$ is the wavenumber corresponding to the largest scale in the system. The equations of motion for u_n and θ_n are:

$$\frac{du_n}{dt} = I_u(k_n) - \nu k_n^2 u_n + \alpha g \theta_n \quad (1)$$

$$\frac{d\theta_n}{dt} = I_\theta(k_n) - \kappa k_n^2 \theta_n + f_n \quad (2)$$

where f_n is the forcing term acting only on the first few shells. The nonlinear terms $I_u(k_n)$ and $I_\theta(k_n)$ are taken to couple quadratically with the nearest shells and sometimes also the next nearest shells, and are constructed to satisfy two conservation laws of energy and entropy (proportional to $|\theta_n|^2$) in the limit of $\nu \to 0$ and $\kappa \to 0$:

$$\frac{d}{dt}\left[\frac{1}{2}\sum_{n=1}^{N}|u_n|^2\right] - \alpha g \sum_{n=1}^{N} Re\{u_n\theta_n^*\} = 0 \quad (3)$$

$$\frac{d}{dt}\left[\frac{1}{2}\sum_{n=1}^{N}|\theta_n|^2\right] = 0 \quad (4)$$

As a result, the nonlinear terms $u_n^* I_u(k_n)$ and $\theta_n^* I_\theta(k_n)$ should have a fluxlike form such that the evolution equations of energy and entropy in the nth shell are:

$$\begin{aligned}\frac{d}{dt}\left[\frac{|u_n|^2}{2}\right] &= F_u(k_n) - F_u(k_{n+1}) - \nu k_n^2 |u_n|^2 \quad (5)\\ &\quad +\alpha g Re\{u_n\theta_n^*\}\end{aligned}$$

$$\frac{d}{dt}\left[\frac{|\theta_n|^2}{2}\right] = F_\theta(k_n) - F_\theta(k_{n+1}) - \kappa k_n^2|\theta_n|^2 + f_n\theta_n^* \quad (6)$$

The fluxes $F_u(k_n)$ and $F_\theta(k_n)$ are respectively the rates of energy and entropy transfer from the $n-1$th shell to the nth shell.

In the Brandenburg model, u_n and θ_n are real variables with [6, 7]:

$$\begin{aligned}I_u^B(k_n) &= ak_n(u_{n-1}^2 - 2u_nu_{n+1})\\ &\quad +bk_n(u_nu_{n-1} - 2u_{n+1}^2) \quad (7)\end{aligned}$$

$$\begin{aligned}I_\theta^B(k_n) &= \tilde{a}k_n(u_{n-1}\theta_{n-1} - 2u_n\theta_{n+1})\\ &\quad +\tilde{b}k_n(u_n\theta_{n-1} - 2u_{n+1}\theta_{n+1}) \quad (8)\\ F_u^B(k_n) &= (au_{n-1} + bu_n)k_nu_{n-1}u_n \quad (9)\\ F_\theta^B(k_n) &= (\tilde{a}u_{n-1} + \tilde{b}u_n)k_n\theta_{n-1}\theta_n \quad (10)\end{aligned}$$

where a, b, $\tilde{a}$ and $\tilde{b}$ are positive parameters. In the SabraT model, u_n and θ_n are complex variables with [11]:

$$\begin{aligned}I_u^S(k_n) &= ik_n\lambda(u_{n+1}^*u_{n+2} - \frac{\delta}{2}u_{n-1}^*u_{n+1}\\ &\quad +\frac{1-\delta}{4}u_{n-1}u_{n-2}), \quad (11)\\ I_\theta^S(k_n) &= ik_n(\alpha_1u_{n+1}^*\theta_{n+2} + \alpha_2u_{n+2}\theta_{n+1}^*\\ &\quad +\beta_1u_{n-1}^*\theta_{n+1} - \beta_2u_{n+1}\theta_{n-1}^*\\ &\quad -\gamma_1u_{n-1}\theta_{n-2} - \gamma_2u_{n-2}\theta_{n-1}) \quad (12)\\ F_u^S(k_n) &= \lambda\mathrm{Im}[k_{n-1}u_{n-1}^*u_n^*u_{n+1}\\ &\quad +(1-\delta)k_{n-2}u_{n-2}^*u_{n-1}^*u_n] \quad (13)\\ F_\theta^S(k_n) &= \mathrm{Im}[\gamma_1(k_nu_{n-1}\theta_{n-2}\theta_n^* + k_{n+1}u_n\theta_{n-1}\theta_{n+1}^*)\\ &\quad -\beta_2k_nu_{n+1}^*\theta_{n-1}\theta_n + \gamma_2k_nu_{n-2}\theta_{n-1}\theta_n^*] \quad (14)\end{aligned}$$

The parameters $\alpha_{1,2}$, $\beta_{1,2}$ and $\gamma_{1,2}$ are determined by

$$\begin{aligned}\alpha_1 = 4\tau\ ,\quad \beta_1 &= 1-\delta-2\tau\ ,\quad \gamma_1 = -\tau,\\ \alpha_2 = 2-4\tau\ ,\quad \beta_2 &= 1-2\tau\ ,\quad \gamma_2 = \tau - \frac{1-\delta}{2} \quad (15)\end{aligned}$$

with three free parameters λ, δ and τ. In particular, we fix $\lambda = 2$ and $\tau = 0.7$ and vary δ. The value $\delta = 1$ is the boundary value separating two families of Sabra model: a family of three-dimensional-like models for $0 < \delta < 1$ and a family of two-dimensional-like models for $1 < \delta < 2$. We focus on $0 < \delta < 1$ in this paper.

We study the scaling behavior of the velocity and temperature structure functions, $\langle|u_n|^p\rangle$ and $\langle|\theta_n|^p\rangle$, with scaling exponents ζ_p and ξ_p defined by:

$$\langle|u_n|^p\rangle \sim k_n^{-\zeta_p}\ ; \qquad \langle|\theta_n|^p\rangle \sim k_n^{-\xi_p} \quad (16)$$

where $\langle\ldots\rangle$ denotes a time average. The K41 scaling would be characterized by $\zeta_p = \xi_p = p/3$ while the BO scaling by $\zeta_p = 3p/5$ and $\xi_p = p/5$. In our numerical calculations, we integrate the equations of motion using fourth order Runge Kutta method with an initial condition of $u_n = \theta_n = 0$ except for a small perturbation of θ_n at intermediate values of n. The Brandenburg model is forced with $f_n = f\delta_{n,0}$ where f is a uniform random noise while the SabraT model is forced with a Gaussian time-correlated noise acting on $n = 3$ and 4 only [12]. For the results presented in this work, we summarize the parameters used in Table I.

In the Brandenburg model, the scaling behavior depends on the relative magnitudes of the parameters a and b, as reported in earlier studies [6]. When b/a is larger than some critical value of about 2, the scaling exponents ζ_p and ξ_p are given by the BO values plus corrections. The scaling behavior improves with b/a. On

TABLE I: Values of the parameters used for the results presented.

Brandenburg model						
a	b	$\tilde{a}$ and $\tilde{b}$	ν	κ	αg	N
0.01	1	1	5×10^{-17}	5×10^{-15}	1	32
0.31	0.6	1	5×10^{-9}	5×10^{-9}	1	25
SabraT model						
δ	λ	τ	ν	κ	αg	N
0.5	2	0.7	10^{-8}	10^{-8}	1	23
0.8	2	0.7	10^{-8}	10^{-8}	1	23

the other hand, when b/a is smaller but close to the critical value, the scaling exponents ζ_p and ξ_p are the same as those obtained in the case of passive scalar advection in which the coupling term $\alpha g \theta_n$ with temperature in the velocity equation of motion is replaced by a random forcing at $n = 0$. This indicates that buoyancy does not play a part in the statistical properties in this case. The scaling exponents for $b/a = 100$ and $b/a = 1.94$ are shown respectively in Figs. 1 and 2. For even smaller values of b/a, further away from the critical value, the system is not chaotic, and in most of the shells the solution is given instantaneously by the fixed-point solution of $u_n = Ak_n^{-1/3}$ and $\theta_n = Bk_n^{-1/3}$, which holds exactly in the limit of large N and $\nu = \kappa = \alpha g = 0$.

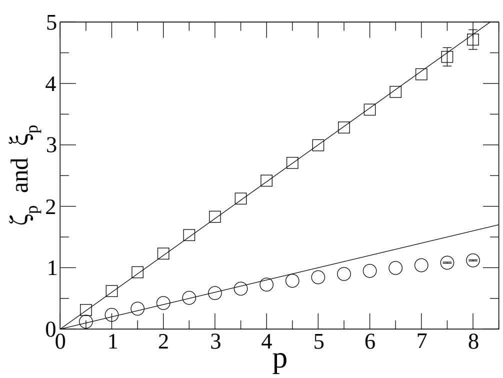

FIG. 1: The scaling exponents ζ_p (squares) and ξ_p (circles) for Brandenburg model with $a = 0.01$ and $b = 1$. The error increases with p and the largest errors are shown. Comparing with the two solid lines of slopes 1/5 and 3/5 shown, it can be seen that the scaling behavior is BO with corrections.

For the SabraT model, we find that the values of ζ_p remain the same as those in the Sabra model without the coupling term $\alpha g \theta_n$ for all the values of δ studied, again indicating that buoyancy does not play a role in determining the statistical properties in the SabraT model for $0 < \delta < 1$. The precise values of ζ_p depend on δ, as was reported in the GOY model [13]. In Fig. 3, we present the results for ζ_p and ξ_p for $\delta = 0.5$, a conventional value at which the model conserves helicity in the inviscid limit [13]. In this case, the values of ζ_p are well described by the She-Leveque result [14] of $\zeta_p = p/9 + 2[1 - (2/3)^{p/3}]$, as was also reported [9] for the GOYT model with $\delta = 0.5$.

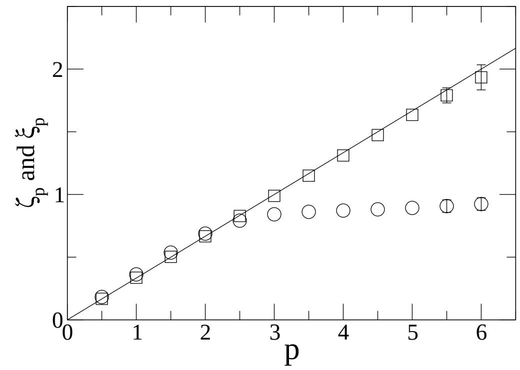

FIG. 2: Same as Fig. 1 for $a = 0.31$ and $b = 0.6$. The solid line shown has slope 1/3.

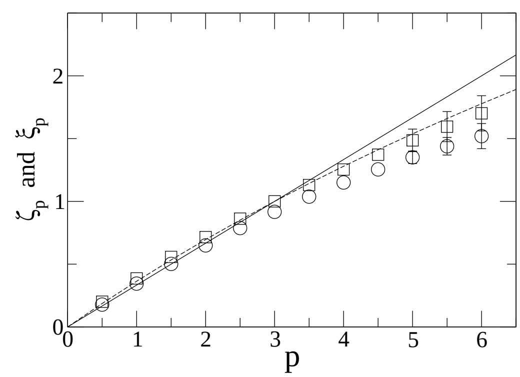

FIG. 3: Same as Fig. 2 for the SabraT model with $\delta = 0.5$. The solid line shown has a slope of 1/3 while the dashed line is the She-Leveque result[14].

III. THE BUOYANCY SCALE

In this section, we discuss how to determine whether buoyancy is relevant or not in determining the statistical properties. Consider Eq. (5), which is the energy budget. The third term of the right hand side is the rate of energy dissipation in the nth shell due to viscosity while the last term is the power injected into the nth shell by the buoyancy forces. It is thus reasonable to take buoyancy to be significant in the nth shell if

$$|\alpha g \langle Re\{u_n \theta_n^*\}\rangle| > \epsilon \tag{17}$$

where $\epsilon \equiv \nu \sum_n k_n^2 \langle |u_n|^2 \rangle$ is the average energy dissipation rate. We denote the scale at which the equality sign in Eq. (17) holds to be the buoyancy scale k_{n^*}. Hence

buoyancy is relevant and significant for $n < n^*$ and irrelevant or insignificant for $n > n^*$. It is easy to show that for u_n and θ_n satisfying exactly K41 or BO scaling, $k_{n^*} = 1/L_B$, where $L_B \equiv \epsilon^{5/4}\chi^{-3/4}(\alpha g)^{-3/2}$ is the Bolgiano length [15] and χ is the average thermal or entropy dissipation rate given by $\chi \equiv \kappa \sum_n k_n^2 \langle|\theta_n|^2\rangle$.

As shown in Figs. 4 and 5, we find that Eq. (17) is satisfied for most of the shells only in the Brandenburg model with b/a larger than the critical value. When b/a is smaller than the critical value, buoyancy is insignificant in all except the largest shells. For the SabraT model, we find that buoyancy is insignificant in all except the largest shells for all the values of δ studied. The results for $\delta = 0.5$ and $\delta = 0.8$ are shown in Fig. 6.

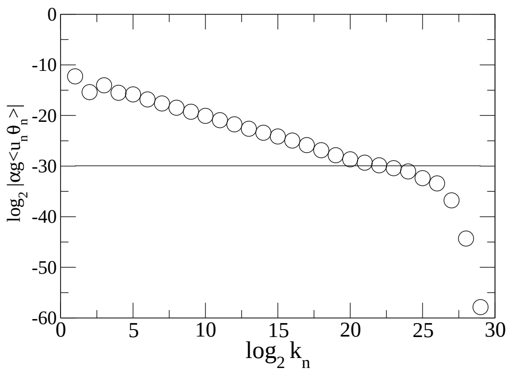

FIG. 4: Comparison of $|\alpha g\langle u_n\theta_n\rangle|$ (circles) with ϵ (solid line) in each shell for the Brandenburg model with a large value of $b/a = 100$.

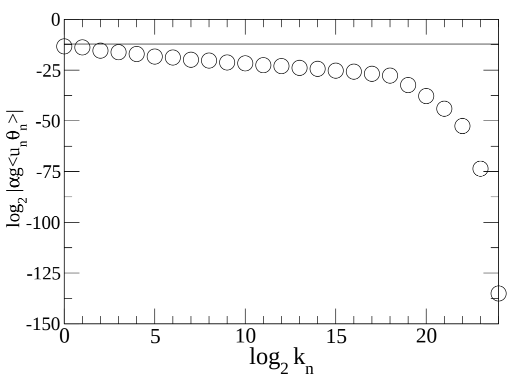

FIG. 5: Same as Fig. 4 for a small value of $b/a \approx 1.9$.

One naturally expects different scaling behavior when buoyancy is significant and when it is not. In this sense, it is not puzzling that different scaling behavior was reported in the the various shell models proposed. Indeed we find BO scaling plus corrections when buoyancy is significant and K41 scaling plus correction when it is not. The two different scaling behavior can be understood by studying the evolution equations of energy and entropy. In the intermediate range where external forcing is not acting and where energy and entropy dissipation rates are both small, Eqs. (5) and (6) can be approximately written as:

$$F_u(k_n) - F_u(k_{n+1}) + \alpha g Re\{u_n\theta_n^*\} \approx 0 \qquad (18)$$

$$F_\theta(k_n) - F_\theta(k_{n+1}) \approx 0 \qquad (19)$$

From Eq. (19), $F_\theta(k_n)$ is independent of k_n in the intermediate range, implying that there is an entropy cascade. From Eq. (18), we see that $\alpha g Re\{u_n\theta_n^*\}$ is comparable with F_u when buoyancy is significant, and $F_u(k_n) - F_u(k_{n+1}) \approx 0$ when buoyancy is insignificant. Thus when buoyancy is insignificant, there is also an energy cascade as in the usual inertia-driven turbulence.

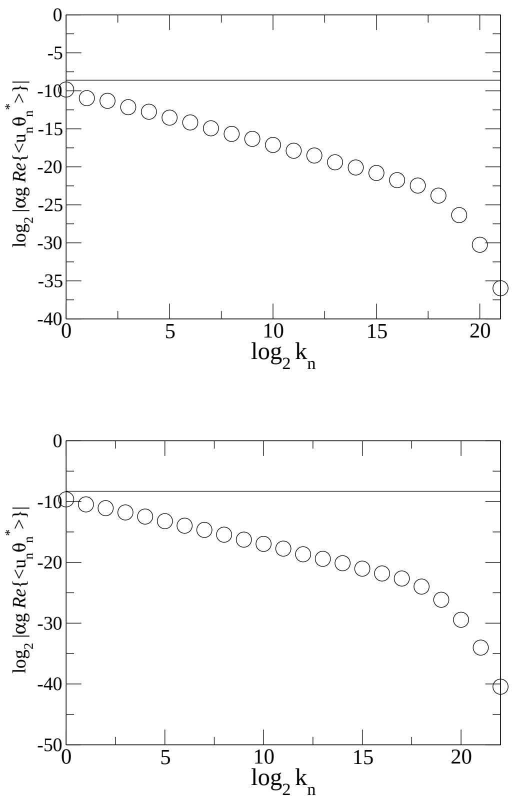

FIG. 6: Comparison of $|\alpha g\langle Re\{u_n\theta_n^*\}\rangle|$ (circles) with ϵ (solid line) in each shell for the SabraT model with $\delta = 0.5$ in the top panel and $\delta = 0.8$ in the bottom panel.

In the case when buoyancy is significant, there is only the cascade of entropy. As a result, one expects the statistical properties to be controlled by the entropy cascade. Specifically, one expects [16] the statistical properties of u_n and θ_n to be determined solely by F_θ, αg, and k_n:

$$|u_n| = \phi_u(\alpha g)^{2/5}|F_\theta(k_n)|^{1/5}k_n^{-3/5} \qquad (20)$$

$$|\theta_n| = \phi_\theta(\alpha g)^{-1/5}|F_\theta(k_n)|^{2/5}k_n^{-1/5} \qquad (21)$$

where ϕ_u and ϕ_θ are dimensionless random variables that are independent of k_n and statistically independent of $F_\theta(k_n)$. On the other hand, when buoyancy is insignificant, there is also the energy cascade. Thus one expects the statistical properties of u_n and θ_n to be determined by F_u, F_θ and k_n:

$$|u_n| = \psi_u |F_u(k_n)|^{1/3} k_n^{-1/3} \qquad (22)$$

$$|\theta_n| = \psi_\theta |F_u(k_n)|^{-1/6} |F_\theta(k_n)|^{1/2} k_n^{-1/3} \qquad (23)$$

where ψ_u and ψ_θ are dimensionless random variables that are independent of k_n and statistically independent of $F_u(k_n)$ and $F_\theta(k_n)$. Hence we have

$$\langle |u_n|^p \rangle \sim \langle |F_\theta(k_n)|^{p/5} \rangle k_n^{-3p/5} \qquad (24)$$

$$\langle |\theta_n|^p \rangle \sim \langle |F_\theta(k_n)|^{2p/5} \rangle k_n^{-p/5} \qquad (25)$$

when buoyancy is significant and

$$\langle |u_n|^p \rangle \sim \langle |F_u(k_n)|^{p/3} \rangle k_n^{-p/3} \qquad (26)$$

$$\langle |\theta_n|^p \rangle \sim \langle |F_u(k_n)|^{-p/6} |F_\theta(k_n)|^{p/2} \rangle k_n^{-p/3} \qquad (27)$$

when it is not. Equations (24) and (25), and Eqs. (26) and (27) thus respectively give BO and K41 scaling plus intermittency corrections for the case when buoyancy is significant and when it is not, just as what was found numerically. Moreover, when buoyancy is significant, the intermittency corrections are solely due to fluctuations in F_θ while in the case when buoyancy is insignificant, the intermittency corrections are due to fluctuations in both and F_u and F_θ. We have checked and verified [16] Eqs. (24) and (25).

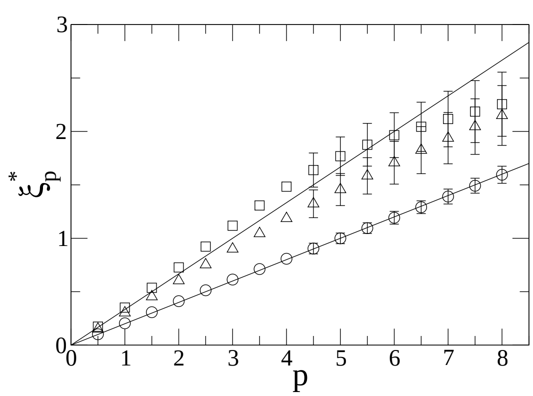

FIG. 7: The scaling exponents ξ^*_p of the conditional temperature structure functions $\langle |\theta_n|^p \mid F_\theta = x \rangle$ for the Brandenburg model with small b/a (squares) and large b/a (circles), and the SabraT model with $\delta = 0.5$ (triangles). The error increases with p and the largest errors are shown. Two solid lines with slopes 1/3 and 1/5 are shown.

Our work shows that the mere presence of a coupling term between velocity and temperature in the equations of motion does not automatically imply that buoyancy is significant and affects the statistical properties. This leads to the question: How can one tell whether temperature is behaving as an active or a passive scalar in models for turbulent thermal convection? For shell models, one can use Eq. (17). If Eq. (17) is satisfied in most shells then buoyancy is significant and temperature is active otherwise temperature would behave as a passive scalar. It would also be useful to have some other criterion that involves directly the statistical features of temperature. Equations (20) and (21) imply that when buoyancy is significant, the conditional statistics of u_n and θ_n at fixed values of F_θ would have simple BO scaling with no corrections [16]. On the other hand, this is not true when buoyancy is insignificant; instead Eqs. (22) and (23) indicate that the conditional statistics of u_n and θ_n at fixed values of F_θ continue to deviate from simple K41 scaling. Hence one can study the conditional statistics of temperature at fixed values of the entropy transfer rate. If these conditional statistics are described by simple scaling then temperature is behaving as an active scalar. Otherwise if the conditional statistics remain anomalous then temperature is behaving as a passive scalar. To check this idea, we calculate the the conditional temperature structure functions at fixed values of entropy transfer rate and their scaling exponents ξ^*_p:

$$\langle |\theta_n|^p \mid F_\theta = x \rangle \sim k_n^{-\xi^*_p} \qquad (28)$$

for the SabraT model the Brandenburg model for both small and large values of b/a. The results of ξ^*_p do not depend on x and are shown in Fig. 7. It can be seen that for Brandenburg model with large b/a, ξ^*_p are indeed well described by the BO values of $p/5$. Also, as expected, for both the Brandenburg model with small b/a and the SabraT model, ξ^*_p's continue to deviate from the K41 values of $p/3$.

IV. CONCLUSIONS

Various shell models have been proposed for turbulent thermal convection. K41 scaling plus corrections has been reported in most of these models while BO scaling plus intermittency corrections is reported in the Brandenburg model with suitable parameters. In this paper, we have shown that buoyancy is not always significant and relevant to the statistical properties in these shell models even though there is an explicit coupling term with temperature in the equation of motion for velocity. We have further clarified that BO scaling plus corrections would be observed only in the shell models in which buoyancy is significant. For shell models in which buoyancy is insignificant, the statistical properties remain the same as in the case in which the coupling term with temperature is absent. We have argued that the statistics properties are controlled solely by the cascade of entropy when buoyancy is significant but controlled by both the cascades of energy and entropy when buoyancy is not significant, and shown how this leads to the two different

scaling behavior in the two cases. We have further shown that the intermittency corrections are solely attributed to fluctuations of the entropy transfer rate when buoyancy is significant but are caused by fluctuations of both the energy and entropy transfer rate when buoyancy is insignificant. As a result, the conditional temperature structure functions at fixed entropy transfer rate would have simple scaling when buoyancy is significant but remain anomalous when buoyancy is insignificant. We have demonstrated how this feature can be used as a criterion to distinguish whether temperature is acting as an active or a passive scalar.

Acknowledgments

This work is supported by the Hong Kong Research Grants Council (CUHK 400304 and CA05/06.SC01).

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